Initiation into Radiotelescopes

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This initiation into radio telescopes is written to make the little error-lobes often shown in images of extraterrestrial radio sources more comprehensible.

Since radar techniques have been developed from the early fifties on , in the beginning mainly to detect planes and submarines, then for navigation purpose and finally for applications (as to measure properties of solids as carrier mobilities etc.) and astronomie, there exists a vast literature about this subject. So this comprehension may also serve as a first, introductionary overview.

We mainly follow the course held by D.Downes, IRAM, Grenoble [1]. The subjects treated are

- ♡ Transfer functions of single dishes [3] and arrays of dishes [4]
- ♡ Beam patterns
- ♡ Antenna and brightness temperature
- ♡ Aperture and beam efficiency
- ♡ Fourier weighting across the aperture
- ♡ Calibration methods
- ♡ Effect of phase errors due to surface irregularities
- ♡ Effect of phase errors due to atmospheric fluctuations
- ♡ Interferometer response
- ♡ Path compensation
- ♡ Sensitivity limits for radio arrays
- ♡ Strategies for Fourier coverage

[2] As done e.g. by the Fraunhofer Institut for Applied Solid State Physics, IAF Freiburg.

[3] A "dish" is evidentally an antenna. For there are terminological terms much worse, we come back to that later.

[4] *Dish arrays*

Introduction

A radio telescope is a wattmeter, measuring the power per unit area per unit frequency, the *flux density* of a source in the sky.

The usual unity is the Janski (Jy):

$$1\,Jy = 10^{-26} W m^{-2} Hz^{-1}$$

Ground based Radiotelescopes can observe within the radio window from several MHz up to 300 GHz. Principally, two different configurations for detection can be distinguished:

single dishes The pattern is formed by a weighting of the Fourier components of the aperture by a feed horn.

arrays Several dishes syntesize an aperture much larger than the physical antenna structure of one dish. This results not only in a higher resolution, but also in a spatially filtering.

Fourier Coverage of the Telescope Aperture

The incomming radiation is spatially filtered by the *radio feed*. The result of this spatially filtering of the aperture function is called the *grading function*. Ideally, it can be seen as the product of the Airy-function with an gaussian, to suppress sidelobes (see f.ex. [2, fig. 11.33, p. 525]).

The grading function $g(x, y)$ specifies the *far field voltage pattern* according to

$$V(l, m) \propto \mathcal{F}\{g(x, y)\} \tag{1}$$

with $l = \sin\theta \sin\phi$ and $m = \sin\theta\cos\phi$. Θ is the angle of incidence relative to the normal plane, Φ is the azimuth from the y-axis.

The *beam pattern* is the *power pattern* in the far field:

$$P(l, m) \propto |V(l, m)|^2 \tag{2}$$

The autocorrelation theorem gives

$$\begin{aligned} \mathcal{F}\{P(l, m)\} &= g \otimes g^* = \int_{-\infty}^{\infty} g(x, y) g^*(x - u, y - v)\, dx\, dy \\ &= W(u, v), \end{aligned} \tag{3}$$

the instrumental transfer function. Hence the transfer function of the instrument is the autocorrelation of the grading function.

Annotations:

- ♡ The beam pattern $= \mathcal{F}^{-1}$ {transfer function}
- ♡ Without spatially filtering and with ideal focussing the beam pattern of an uniformly illuminated circular aperture would be the Airy pattern.

♡ In practice, the incomming radiation is not perfectly focused.

Spatially filtering is done with monomode radio feeds, where the waveguide mode is spatially nearly gaussian. For there is no sharp cut-off at the edges, the sidelobes are not ideally suppressed.

The radio feeds, or "feed antenna", are the collecting device at the focus. They can either be placed in the primary focus or, in a Cassegrain configuration, in the secondary focus [3, fig. 3.22, p. 74]. Due to the scattering at the feed, the feed receives also energy from angles not intercepted by the parabolic reflector. This so-called "spillover" comes in the Cassegrain secondary focus configuration mainly from the sky, which is colder than the ground at most radio-wavelengths.

Annotations:

♡ In both configurations (primary or secondary focus) holds: The broader the gaussian grading, the more uniform the aperture illumination, but the lower the aperture efficiency.

♡ Non-gaussian illumination may be obtained with shaped sub-reflectors or lenses.

In return for low sidelobes and low spillover, main lobes broader than that of the Airy pattern are accepted: The feed horn typically has -10 to -14 dB at the edge of the dish, with a gaussian shape in the intermediate region.

The far field pattern then is nearly gaussian with a FWHP (full-with to half-power) of

$$\Theta_b \simeq 1.2 \frac{\lambda}{D} \text{ [rad]} \tag{4}$$

corresponding to a main beam region which contains the parts of the antenna pattern directly concerned with the source to be measured of

$$\Omega_b \simeq 1.133\ \Theta_b^2 \text{ [sterrad]} \tag{5}$$

where D is the dish diameter and λ is the wavelength.

Annotations:

♡ Grading also is applied in the data reduction to reduce sidelobes of the syntesized beam. The beamwith obtained therefore is narrower than the beamwith for a graded equivalent single dish.

♡ Eq. 5 gives the solid angle of an equivalent gaussian with same FWHP good to 5%.

A better aproximation (good to 1%)can be obtained using the FW to one-tenth-power Θ_{-10dB}:

$$\Omega_b \simeq 0.3411(\Theta_{-10dB})^2$$

♡ If the source is gaussian with FWHS (S = signal): Θ_s, the response pattern will also be gaussian with FWHS Θ_r:

$$\Theta_r^2 = \Theta_s^2 + \Theta_b^2$$

Practical calculations on the power recieved are done without the explicite knowledge of the grading function. Therefore we introduce some concepts in the following paragraphs.

Antenna Temperature and Brightness Temperature

A contribution of a source in the sky to the background can be measured while measuring the

source + background

and the background

expressed either in T_a or T_b:

Antenna temperature T_a: T_a is the temperature of an equivalent resistor which would give the same power per unit bandwith p (in WHz^{-1}), as measured at the output of the instrument, as the celestrial source. The power therefore is related to the temperature by the Nyquist formula

$$p = k_B\, T_a.$$

Brightness temperature T_b: T_b is the Rayleigh-Jeans temperature of an equivalent black body which would give the same power per unit solid angle and frequency as the celestrial source.

The radio brightness (flux density per unit solid angle) is therefore related to the brightness temperature by the Rayleigh-Jeans law:

$$B = 2k_B T_b/\lambda^2$$

Annotations:

♡ T_a is antenna specific, whereas T_b is a property of the source.

♡ T_a, T_b allow to calculate the power recieved by single dishes without knowing the details of the Fourier coverage of the aperture.

Grading consequences are taken into account through the aperture efficiency and the beam efficiency:

Aperture Efficiency and Beam Efficiency

Aperture Efficiency ϵ_{ap}:

$$\epsilon_{ap} = \frac{A_e}{A} \tag{6}$$

with the effective collecting area A_e and the geometric area A.

Effective Beam Efficiency B_{eff}:

$$B_{eff} = \frac{T_a'}{T_{mb}} \tag{7}$$

with the antenna temperature corrected for atmospheric extinction, $T_a' = T_a e^{\tau_\nu}$, and the main beam brightness temperature T_{mb}

It is

$$B_{eff} = \epsilon_{ap} \frac{A\,\Omega_b}{\lambda^2},$$

where in the case of a single dish, $A = \pi D^2/4$:

$$B_{eff} = 0.8899 \left(\frac{\Theta_b}{\lambda/D}\right)^2 \epsilon_{ap} \tag{8}$$

with the beamwith Θ_b in rad.

Annotations:

- Eq. approximates the integral over the beam. Therefore the main-beam efficiency can be calculated to the same precission as one can measure aperture efficiency.
- A better approximation is

$$B_{eff} = 0.2679 \left(\frac{\theta_{-10dB}}{\lambda/D}\right)^2 \epsilon_{ap}$$

- Measurement of ϵ_{ap} can be done via a celestrial source of a known flux density. The antenna temperature therefore is obtained by a comparison with a noise standard.
- Measurement of B_{eff} can be done via a planet of the same angular diameter than the beam, assuming the true temperature. The correction relies on an assumed gaussian beamshape.

The contribution of the source only is then obtained by substracting the measurement of the background from the measurement of the source and the background.

Noise

The electronical equipment behind the feed antenna, that is

- a low-noise pre-amplifier (not for milimeter-waves)
- a mixer
- some IF-amplifiers
- a backend detector
- an AD-converter

imposes the receiver noise. The instrument then can be described by a system temperature, whitch is the temperature of an equivalent resistor with the same noise power as the entire system as follows:

Receiver Temperature T_R: T_R is given within a frequency intervall $\Delta\nu$ by the Nyquist formula:

$$p = k_B T_R.$$

It comes from the mixer plus the IF-stages with conversion loss L according to

$$T_R = T_M + L T_{IF}$$

System Temperature T_{sys}: T_{sys} is defined as

$$T_{sys} = T_R + T_{sky} + T_{ground} \tag{9}$$

where

$$T_{sky} = F_{eff} T_{amb}(1 - e^{-\tau_\nu}) \tag{10}$$

$$T_{ground} = (1 - F_{eff}) T_{amb} \tag{11}$$

with the ambient temperature, T_{amb}, same for earth and ground here, the receiver temperature T_R and the forward efficiency of the antenna F_{eff}.

The r.s.m. sensivity ΔT_a is

$$\Delta T_a = T_{sys}(\Delta\nu\tau)^{-0.5}$$

with the bandwith per observing channel $\Delta\nu$ and the integration time τ. This may be multiplied with a factor depending on the observing mode.

Chopper Wheel Calibration

In chopper wheel calibration, the sky itself is used as calibration source. The calibration signal will be the difference between an absorber at ambient temperature and the sky.

We consider the simplified case of single-sideband observing (image-band suppressed) and $T_{atm} \approx T_{amb}$. The signals are

$$V_{amb} = G\,(T_{amb} + T_R) \tag{12}$$

$$V_{sky} = G\,[F_{eff} T_{sky} + (1 - F_{eff}) T_{ground} + T_R] \tag{13}$$

$$= G\,[F_{eff} T_{amb}(1 - e^{-\tau_\nu}) + (1 - F_{eff}) T_{amb} + T_R]$$

where G is the varying factor to be calibrated out.

The calibration signal therefore is

$$V_{cal} = V_{amb} - V_{sky} = G\,F_{eff}\,T_{amb}\,e^{-\tau_\nu} \tag{14}$$

The signal from the source is

$$V_{sig} = G\,T_a' e^{-\tau_\nu} \tag{15}$$

$$T_a' = \frac{V_{sig}}{V_{cal}} F_{eff}\,T_{amb} \tag{16}$$

or

$$T_a^* = \frac{T_a'}{F_{eff}} = \frac{V_{sig}}{V_{cal}} T_{amb} \tag{17}$$

where T_a' is the antenna temperature of the source outside the atmosphere, $\tau_\nu = \tau_0 \sec z$, τ_0 = zenith optical depth, and z = zenith angle.

Annotations:

♡ T_a^* is corrected for atmospheric attenuation.

♡ T_a^* is a fictional temperature, convenient only for the chopper wheel method.

It can be thought of as a "forward beam brightness temperature", that is a brightness temperature of an equivalent source which fills the entire 2π steradians of the forward beam pattern.

♡ Taking into account a gain factor g_i of the image sideband relative to that of the signal sideband ($g_s \equiv 1$) and an average temperature of the atmosphere T_{atm} differing from the temperature of the chopper, one has

$$T_a^* = \frac{V_{sig}}{V_{cal}} T_{cal}$$

with

$$T_{cal} = (T_{amb} - T_{atm})(1 + g_i)e^{\tau_s} + T_{atm}(1 + g_i)e^{\tau_s - \tau_i}$$

with τ_s, τ_i = atmospheric opacities in signal and image bands.

♡ The chopper wheel method does not correct antenna temperature for all telescope losses, in particular not for losses due to surface irregularities.

Phase Errors due to Surface Irregularities

A wavefront distortion of an r.m.s. (root mean square) surface tolerance Δz induces an r.m.s. phase error

$$\Delta\Phi = 4\pi\left(\frac{\Delta z}{\lambda}\right),$$

therefore a phase error factor $e^{i\Delta\Phi}$ to the electric field at the focus of the dish is induced.

The main-beam power pattern P_m is reduced to its error-free value P_0 for small $\Delta\Phi$, (i.e. $\Delta\Phi = 1 + i\Delta\Phi - 0.5(\Delta\Phi)^2$, $\Delta\Phi \ll 1$) by

$$\frac{P_m}{P_0} = 1 + \langle\Delta\Phi\rangle^2 - \langle\Delta\Phi^2\rangle$$

For symetrically distributed irregularities is $\langle\Delta\Phi\rangle = 0$, hence

$$\frac{P_m}{P_0} = 1 - \langle\Delta\Phi^2\rangle$$

Expressing $e^{i\Delta\Phi}$ by cosine and sine components, the power reduction in the direction of maximum response is

$$\frac{P_m}{P_0} = \cos^2\Delta\Phi$$

Annotations:

♡ In the case of gaussian, uniformly distributed errors over the apperture with correlation length l_c satisfying $\lambda \ll l_c \ll D$,

$$\frac{P_m}{P_0} = e^{-\Delta\Phi^2} = e^{-(4\pi\Delta z/\lambda)^2}$$

♡ e.g. the IRAM telescope, $\lambda = 30\ cm$ at $230\ GHz$:

$$\Delta\Phi = \left(-\ln\frac{\epsilon_{ap}(230\,GHz)}{\epsilon_{ap}(90\,GHz)}\right)^{0.5} \quad (18)$$

$$= \left(-\ln\frac{0.27}{0.50}\right)^{0.5} = 0.78\ \text{rad} \quad (19)$$

corresponding to

$$\Delta z \simeq 80\ \mu m$$

In a statistical description, the power is removed into an error beam of wit~~h~~dth

$$\Theta_e \approx \lambda/l_c$$

and

$$\frac{P_e}{P_0} = 1 - \frac{P_m}{P_0} = 1 - e^{-\Delta\Phi^2}$$

Annotations:

♡ The error pattern for large phase errors, where the main lobe disappeared, is

$$P_e(\Theta) = \left(\frac{\pi l_c}{\lambda}\right)^2 \frac{[1 - e^{-\Delta\Phi^2}]}{\Delta\Phi^2} \exp\left[-\left(\frac{l_c \sin\Theta}{8\Delta z}\right)^2\right]$$

with a FWHP of

$$\Theta_e = 13.3\,\frac{\Delta z}{l_c}$$

independent from λ.

♡ One can define an angle analogous to the seeing disk in optical interferometrie, where the phase error $\Delta\Phi = 1$ rad, then the FWHP is

$$\Theta_e(\Delta\Phi = 1) = 1.06\,\frac{\lambda}{l_c}$$

Therefore the correlation length l_c is analogous to the Fried parameter r_0 in optical seeing.

The error beam is analogous to the seeing disk.

♡ In presence of the error pattern, the beam efficiency, B_{eff}, must be interpreted as full-beam efficiency (main-lobe plus error pattern). T_{mb} becomes a beam-averaged brightness temperature, containing the error pattern.

Phase Errors due to Annomalous Refraction

In the neutral atmosphere, at 0 to 30 GHz, away from resonances the refractivity is

$$\begin{aligned} N = (n-1)10^6 &= \frac{77.6}{T_{atm}}\left(P + \frac{4810\,e}{T_{atm}}\right) \\ &\equiv N_{dry} + N_{wet} \end{aligned} \qquad (20)$$

with the total atmospheric pressure in millibar P, the partial pressure of water vapor e (10...30 mb at sea level) and the index of refraction n. The temperature of the atmosphere is usually $T_{atm} \sim 280\ K$.

Annotations:

♡ $N_{dry} \propto e^{h/8km}$ can be predicted from hydrostatic equilibrum

♡ For typically $N_{dry} \sim 280$, the *excess path length*, relative to the path of the rays in free space is

$$L_{dry} = 10^6 \int N_{dry} dl \;\simeq 225\ cm$$

♡ $N_{wet} \approx e^{h/2km}$, the '$\approx$'-sign therefore refers to the fact, that H_2O in the atmosphere is not well mixed.

♡ For $N_{wet} \sim 100$, the excess path length is

$$L_{wet} = 10^6 \int N_{wet} dl \simeq 20cm$$

♡ The excess path length due to refraction in the ionosphere is

$$\begin{aligned} L &\sim 10cm\,(\nu/10\,GHz)^{-2} \\ &\sim 0.1cm\,(\nu/100\,GHz)^{-2} \end{aligned}$$

Annomalous refraction is caused by variations in N_{wet} arising from changes Δe in the partial pressure of water vapor which changes the optical path length by 0.5 mm on baselines of 30...100 m.

Annomalous refraction has particular strong consequences for mm and sub-mm telescopes and arrays because of their smaller primary beams: Radio sources move up to 40" with up to 30 Hz, an effect which is stronger at the afternoon and weaker in winter on cold sites.

Annotations:

♡ In contrary to radio interferometrie, in optical interferometrie phase variations are due to the dry component N_{dry}.

♡ Associated variations in the water vapor are less than 0.1 mm, therfore changes in atmospheric opacity of the sky brightness are hardly noticeable.

♡ The phase noise is increased and fringe amplitudes are reduced as the sources move out of the narrow primary beams.

♡ Changes in the optical path length are

$$\Delta l \simeq B^{0.3},$$

B being the baseline.

♡ Single dish data suggests that one is seeing individual packets of moist air rather than many packets at varying distances from the telescope.

From eq. 20 follows, that $\Delta N_{wet} \approx 5\Delta l$. With $T_{atm} = 280\ K$ and a relative humidity of 50 %, the partial pressure of water vapor e is 4.5 millibar. Therefore 20% fluctuation in the relative humidity corresponds to

$$\Delta e \approx 1\ mbar, \quad \Delta N_W \approx 5.$$

Over $L \sim 100$ m, the typical variation in electrical pathlength is

$$\Delta l \approx \Delta N_{wet} 10^{-6} L \sim 0.5\ mm,$$

corresponding to a change in the phase

$$\Delta\Phi = 2\pi(\Delta l/\lambda)$$

The Fourier component corresponding to the diameter D (in the case of an array D is the distance between the dishes) is λ/D. One cycle is a phase change of 2π. The fluctuation in the optical path Δl therefore corresponds to an apparent position shift $\Delta\Theta$:

$$\Delta\Theta = \frac{\Delta\Phi}{2\pi}\frac{\lambda}{D} = \frac{\Delta l}{D}$$

Typically $D = 30$ m, $\Delta\Theta \sim 3"\ldots 5"$.

Beam Switching

If the variations in water vapor are strong and short-term, the sky brightness variations may exceed the system noise. This presence of atmospheric noise can be improved by using beam switching, with the beams having as small a separation as possible.

e.g. $\lambda = 2$ cm, $D = 100$ m, FWHS $\lambda/D \approx 1'$, then a beam switching over $8'$ corresponds to a seperation of ~ 7 m in packets of water vapor located 3 km from the telescope.

The true map T of a source distribution S measured with a beam pattern B will be obtained by convolving the observed map M with a restoring function R, which Fourier transform is given by $1/\hat{C}$, $\hat{C}$ being the beam switching function sampled at intervalls $1/x$ (x = mapping size) as follows:

$$\begin{aligned} T &= S \star B \quad \text{convolution} \\ M &= T \star C \\ \hat{M} &= \hat{T} \cdot \hat{C} \quad \text{in Fourier space} \end{aligned} \qquad (21)$$

hence

$$
\begin{aligned}
\dot{T} &= \dot{M}/\dot{C} \equiv \dot{R} \cdot \dot{M} \\
T &= R \star M
\end{aligned}
$$

Annotations:

- ♡ R is a comb with only a few elements
- ♡ the problem is not ill-posed
- ♡ particular useful in mm, sub-mm and near infrared ranges.

Interferometer Response

The van Cittert - Zernicke theorem correlates the complexe spatial coherence ("response" in the following) R with the real and non-negative source brightness distribution $b(x,y)$. In the case of rectangular coordinates, the result is the *response*

$$
\begin{aligned}
R(u,v) &= \int b(x,y) e^{i2\pi(ux+vy)}\, dx\, dy \qquad (22) \\
b(x,y) &= \int R(u,v) e^{-i2\pi(ux+vy)}\, du\, dv
\end{aligned}
$$

with the *baseline coordinates* (u,v) and the *sky coordinates* (x,y).

In the case of two antennas, the output voltages V_1 and V_2 are cross-correlated as a function of time τ:

$$R(\tau) = \frac{1}{T}\int_0^T V_1(t) V_2^*(t-\tau) dt \qquad (23)$$

For example, a plane wave

$$R(\tau) = Ae^{i\Phi(\tau)} \qquad (24)$$

with the fringe phase

$$\Phi(\tau) = \frac{2\pi D}{\lambda}\cos\Theta(\tau) \qquad (25)$$

and the fringe amplitude A. Θ is the angle of the source measured from the baseline direction. R is maximal for

$$\Phi_{max}(\tau) = n\,2\pi \qquad (26)$$

The spacing between the maxima ('fringe angular spacing') gives the resolution of the interferometer:

$$
\begin{aligned}
\frac{d\Phi}{d\Theta} &= \frac{d}{d\Theta}\left(\frac{2\pi D \cos\Theta}{\lambda}\right) \qquad (27) \\
&= \frac{-2\pi D_p}{\lambda}
\end{aligned}
$$

where $D_p = D\sin\Theta$, the projected baseline. The fringes are therefore seperated on an angular scale of

$$\Delta\Theta = \frac{\lambda}{D_p} \qquad (28)$$

Hence the interferometer is measuring the flux in the Fourier components of the source brightness at spatial frequency D_p/λ_c.

Annotation:

- ♡ We can define a fringe visibility

$$V(u,v) = \frac{\int b(x,y)e^{i2\pi(ux+vy)}dx\,dy}{\int b(x,y)dx\,dy}$$

with $0 \le |V| \le 1$.

Fringe Stopping

Signals received at any pair of antennas are fed to two seperate correlators, which are phase-delayed to each other of $\pi/2$: One signal is the *sine component*, the other is the *cosine component* of the interferometer response.

In both, sine and cosine components of an array of antennas and their corresponding correlators, time-delays are inserted to compensate the difference in the path to one of the antennas due to position (allowing earth rotation) in the following way:

The path compensation is done equivalent to placing the individual antennas of an array on a giant, imaginary paraboloid, pointing at the position of the *phase reference center*.

Alternativly, one may think of the phase delays as placing the antennas on a plane perpendicular to the phase reference center to the sky, with equal length cables to the correlators.

For a plane wave again we have

$$R(\tau) = Ae^{i\Phi}$$

with

$$\Phi = \frac{2\pi D}{\lambda}\cos\Theta$$

The phase delays set $\Phi = 0$ for $\Theta = \pi/4$. Θ_0 being the direction of the imaginary disk to the phase reference center in the sky, the angle to the source for each dish now is measured in respect to Θ_0 according to $\Theta = \Theta' - \Theta_0$ where Θ' is the angle to the source measured from the baseline.

Annotations:

- ♡ The modified fringe plane will be zero for a source component at position Θ_0. The fringes have been "stopped"[1]

[1] Confronted with such a terminologie, we cite J.W.Goethe "Worte sind Schall und Rauch".

- *Phase* usually refers to the residual fringe phase after the fringes "have been stopped". Zero phase corresponds to the position of the phase reference center.
- A source component located at a small angle α from the reference center has a residual phase

$$\Phi = \frac{2\pi D}{\lambda}\cos(\Theta_0 + \alpha) \simeq \frac{2\pi D}{\lambda}\alpha$$

- It is

$$|R| = A = \left(R_{cos}^2 + R_{sin}^2\right)^{0.5}$$

the fringe amplitude and

$$\Phi = \tan^{-1}\frac{\Im R}{\Re R} = \tan^{-1}\frac{R_{sin}}{R_{cos}}$$

is the residual fringe phase corresponding to the position of the relevant source component.
- The source brightness distribution can be written with α as defined above:

$$b(\alpha, \hat{S}) = \int A_D e^{i\Phi} \exp\left(i2\pi\frac{\bar{D}\hat{S}}{\lambda}\right) d\left(\frac{\bar{D}}{\lambda}\right)$$

with the baseline vector $\bar{D}/\lambda$ and the direction of the source $\hat{S}$. Note that if $A = 1$ for all Fourier components (all baselines) for $\Phi = 0$, then $b = \delta(\Theta)$, that is a $1 - Jy$ (Jansky) source located at the phase reference center.

Delay Tracking: The Coherence Length

The averaged output of an interferometer with bandwith $\Delta\nu$ is

$$R = \int_{\nu_1}^{\nu_2} V_1 V_2 \cos\frac{2\pi\nu l}{c}\, d\nu \tag{29}$$

with the path difference l. It goes $R \to 0$ for $c/\Delta\nu = l \equiv l_c$, the *coherence length.*

A pair of antennas can detect fringes as long as the cross correlation performed on the radiation arriving at the two antennas is done within the coherence length.

The sensitivity of an interferometer consisting of an array of antennas is therefore limited to where the path difference corresponding to the angular displacement is greater than the coherence length.

Annotations:

- One can equally define a *coherence time* Δt within a bandwith $\Delta\nu$: $\Delta t = 1/\Delta\nu$. Then $l_c = c\Delta t$.
- The angular range l_c/D is called "delay beam".
- Sources within the field λ/d of the antenna can be resolved if the delay beam is larger than the primary beam of the array antennas of size d

- Planning observations, one tries to have

$$\frac{\lambda}{D} < \frac{\lambda}{d} < \frac{l_c}{D}$$

where especially for astrometrie and accurate stellar diameter measurements the last term is demanded to overcome by a factor 16.

Aperture Synthesis

Aperture synthesis is just *Fourier sampling* as follows:

For a virtual giant dish with dishes at $1 \ldots n$ and voltages $V_1 \ldots V_n$, the detector output power is

$$\left(\sum V_i\right)^2 = \sum V_i^2 + \sum V_i V_j \cos(\Phi_i - \Phi_j) \tag{30}$$

The information is contained in the second term, therefore the cross product describes the interference. To synthesize a large aperture, only $V_i V_j$ must be measured, e.g. in the following way:

1. Synthesizing an image on short time scales: "snapshot" mode or
2. Sample the interferometer output at a number of positions within an aperture of size D, while displacing the antennas.
3. To change the orientation of the baselines, the earth 's rotation can be used, called "earth rotating synthesis".

Sensitivity of an Aperture Synthesis Telescope

To get the sensitivity limit of radio synthesis, we have to consider some fluctuations:

r.m.s. fluctuation in antenna temperature:

$$\Delta T_a = \frac{f\, T_{sys}}{\sqrt{t\Delta\nu}} \tag{31}$$

with the system temperature T_{sys}, the integration time t, the bandwith $\Delta\nu$ and an AD noise factor f.

r.m.s. fluctuations in flux density:

$$\Delta S = \frac{2k_B \Delta T_a e^{\tau_\nu}}{A_e\sqrt{2}}$$

with the atmospheric opacity τ_ν, the effective collecting area $A_e = \epsilon_{ap}\pi D^2/4$ of a single dish of diameter D and the aperture efficiency ϵ_{ap}.

For an array of n identical dishes,

$$N = n(n-1)/2$$

is the number of baselines. Therefore the

r.m.s. fluctuation in flux density is:

$$\Delta S = \frac{2k_B f T'_{sys}}{A_e\sqrt{2Nt\Delta\nu}}$$

where $T'_{sys} = T_{sys}e^{\tau_\nu}$.

For a synthesized beam of solid angle Ω_b, the main-beam brightness temperature T_b is defined by

$$S \equiv \frac{2k_B}{\lambda^2}T_b\Omega_b$$

For an aperture synthesized map the

r.m.s. variation in brightness temperature is:

$$\Delta T_b = \frac{\lambda^2 f T'_{sys}}{A_e\Omega_b\sqrt{2Nt\Delta\nu}} \tag{32}$$

Annotations:

♡ The FWHP of the synthesized beam in case of weighting the data with a taper to -6 dB at the longest baseline length D_{max} is

$$\Theta_b = 0.7\frac{\lambda}{D_{max}}$$

♡ In the case of a circular gaussian synthesized beam:

$$\Omega_{mb} = 1.133\,\frac{0.7\lambda}{D_{max})^2}$$

♡ e.g. a 2-bit, 3-level correlator with $f = 1.23$ leads to

$$\Delta T_b = \frac{2.0\,T'_{sys}D^2_{max}}{\epsilon_{ap}D^2\sqrt{Nt\Delta\nu}}$$

Note $\Delta T_b \propto D^2_{max}$

♡ If data is smoothened to keep the velocity resolution ΔV constant it is:

$$\Delta\nu[Hz] = 10^6\frac{\Delta V[km/s]}{\lambda[mm]}$$

$$\Delta T_b = \frac{42\lambda^{2.5}T'_{sys}}{\epsilon_{ap}D^2\Theta_b^2\sqrt{Nt\Delta V}}$$

where the temperatures ΔT_b, $\Delta T'_{sys}$ are measured in Kelvin, D is measured in meter and Θ_b in arcsec.

The sensitivity often is limited not by the system noise, but by the effects as the following:

1. Incomplete sampling in the Fourier transform (u, v)-plane.
2. The Fourier response to a regular spacing of the antennas, called "grating rings".
3. Aliasing (from "alias"= other name) which is an overestimation of frequencies near the Nyquist frequency because of contributions due to interpolating the data onto a grid to allow Fast Fourier Transformation.

Restoring algorithms to overcome these effects are MEM, CLEAN.

Very Long Baseline Interferometrie (VLBI)

By videorecording techniques the baselines can be extendet to 10^3 to 10^4 km. In order to preserve fringe patterns, phase-drifts and random jumps $\delta\nu$ in the frequencies of the clocks [6] must be less than $1/2\pi$ cycles during the integration time t_{int}:

$$\delta\nu\; t_{int} < \frac{1}{2\pi} \tag{33}$$

hence a relative stability of

$$\frac{\delta\nu}{\nu_0} < \frac{1}{2\pi\nu_0\; t_{int}} \tag{34}$$

is demanded.

e.g. $t_{int} \sim 10^2 s$, $\nu_0 = 10^{10}Hz$:

$$\frac{\delta\nu}{\nu_0} < 2\cdot 10^{-13}$$

can be achieved with a Hydrogen maser with $\delta\nu/\nu_0 \sim 10^{-14}$ over 100 Hz^{-1}, drift $< 1\mu s/yr$ and can be used up to 1000 GHz.

For VLBI, baselines must be measured accurately. Once known with an accuracy of e.g. 1 ft. one can measure

♡ variations in earth's rotation

♡ polar motion down to 1 ft.

Closure Phase

Due to atmosphere effects, in VLBI as in optical interferometrie the wavefront phase is lost.

One can reconstruct the image of the source only if it is possible to recover the Fourier phases; This can be done if fringes can be detected on a sufficiently large number of baselines:

Around a closed triangle of interferometer baselines, the *closure phase*

$$\Phi_{123} = \Phi_{23} + \Phi_{12} + \Phi_{31}$$

is the sum of phases of the source Fourier components. It is a property of the source; Phase errors due to atmosphere and antennas cancel out.

[6] In Radioastronomie, clocks are used not to measure time but frequencies.

Annotations:

- ♡ If n is the number of the antennas in the array, the number of closure phases is $(n-2)/n$.
- ♡ In the case of the Very Large Array (27 dishes), 92% of the phase information can be regained.
- ♡ A generalization of the method is bi-spectrum analysis.
- ♡ A method applying closure phase is Hybrid mapping (Readhead et al. 1980, Nature).

Self-Calibration

Each element in an array contributes an antenna gain factor g_i, with

$$g_i = a_i e^{i\Delta\Phi_i}$$

where $a_i = e^{-\tau_\nu}$ is the amplitude due to atmospheric attenuation and $\Delta\Phi_i$ is the phase error caused by atmosphere or the instrument.

The aim of the calibration is to calibrate the gain factors out. This can be done e.g. as follows: A point source (initial model) is Fourier transformed to the visibility plane. Using the closure phase information, the fringes of the model are updated with an CLEAN or MEM algorithm. The new model then is taken as the initial model.

Annotations:

- ♡ Successive iterations solve for all of the gain factors.
- ♡ A triply nested procedure, consisting an fringe-fitting procedure, the self calibration and the restoring routine CLEAN or MEM, was proposed by Schwab and Cotton (1983).

Strategies for Fourier Coverage

To get a good coverage of the (u,v)-plane, a number of strategies have been developed, depending whether sensitivity or image quality is critical:

1. If sensitivity is critical and the size of the dishes is smaller than the coherence length, then the size should be maximized.
2. If high-quality images are desired and if the angular sizes of the sources are comparable or larger than the field of view of the array antennas, the number of dishes should be maximized.
3. The holes in the (u,v)-plane should be minimized while properly placing the dishes, e.g. by the method of "crystalline arrays" by Cornwell (1986,88).

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